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Proposing Adomian Decomposition Method to Treat two Dimensional Inhomogeneous Mixed Volterra- Fredholm Integral Equation

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***Abstract* — in this project, for the first time reformulating and applying the Adomian decomposition method to treat the numerical solution of two dimensional inhomogeneous mixed Volterra- Fredholm integral equation of the second kind. Two new theorems are proved, which provide and prove the convergence of the method. Tables distributing the comparison of numerical results . For the criterion of the comparison, we compute the least square errors and the running time for the program. Finally, for proposing the technique two examples are solved for demonstrating the validity and applicability. The comparison make was made of results for different iterations. The squares least-square error and running time of the program are written in tables for supporting the method.**

Keywords—Adomian decomposition technique, two-dimensional integral equations, and mixed Volterra- Fredholm integral equation.

I. INTRODUCTION

In the last decades, applied sciences had important role in scientific investigation, because different problems of integral equations were constructed from various applied sciences. This equation in space-time has a very significant role in technological and mechanical problems. Especially types of the integral equation is known as mixed Volterra-Fredholm integral equations [5], [11].

The two-dimensional integral equations (TDIE) is an active subject in applied sciences because varies phenomena in mathematical physics, the diffusion theory, heat conduction theory, electromagnetic waves and engineering problems changed to two-dimensional integral equations [3],[15], for example Remain Hilbert problem can be reformulation or changed to the TDIE , moreover many physical problems can be represented as TDIE, like the volume integral has been replaced by double integral over the domain of the sources in XY-plane, the modeling of inhomogeneous dielectric cylinders and the plane wave scattering problems by an airframe stricter can be changed to TDIE [12]. The two-dimensional inhomogeneous mixed Volterra- Fredholm integral equation (TDIMVFIE) appears in various problems like the ordinary differential equations for two variables on the bounded region [7],[11].

The analytic solution to these problems is usually very difficult to produce. So, for these cases to find numerical solutions to this problem the numerical methods are more practical, because in scientific investigations the numerical techniques are very important for solving these problems numerically. So, using numerical methods with the advent of computers and programming these methods, has been popularized and more importantly [1], [6].

Recently, studying the approximate solutions for integral equations is very important many scientists focused on this point. They used various techniques for this aim [4, 9]. Such as Hacia in 2002 studied Volterra- Fredholm integral equations of the second type (VFIE-2). Abdou in 2005 found a numerical solution for mixed type of Fredholm Volterra integral equation. Rahmani in 2007 solved VFIE-2. Tenwich in 2012 solved one and two dimension Integral equations. Zarebnia in 2014, solved nonlinear Volterra- Fredholm integral equations. Sulaiman in 2016 solved the system of VFIE-2 by using the Adomian decomposition method (ADM). Hasan in 2019 used the Iterative Kernel Method for solving two Dimension Linear Mixed Volterra- Fredholm Integral Equation of the Second Kind. For extending these works, reformulating and using ADM for treating TDIMVFIE for finding the approximate solution of it.

Definition (1): The two dimensional integral equation of the form

$$z(y,t)=g(y,t)+\lambda\int_a^t\int_a^b G(y,t,v,w)z(v,w)dvdw \qquad (1)$$

is called TDIMVFIE, such that $g(y,t)$ is given continuous functions on $H=\{(y,t);a\le y\le b,a\le t\le k_1\}$ and $G(y,t,v,w)$ is given continuous function on the connected region $F=\{(y,t,v,w);a\le v\le y\le b,a\le w\le t\le k_1\}$, and $z(y,t)$ is the unknown function [5],[6].

II. SOLVING THE PROBLEM BY USING ADM.

The main idea of this work, is to reformulate ADM for solving TDIMVFIE. Suppose that the approximate solution of "(1)" be as the form

$$z(y,t)=\sum_{i=0}^{n} z^i(y,t) \qquad (2)$$

Substuting it in "(1)" we get

$$\sum_{i=0}^{n} z^i(y,t)=v(y,t)+\lambda\int_a^t\int_a^b G(y,t,v,w)\sum_{i=0}^{n} z^i(v,w)dvdw \qquad (3)$$

By equaling the powers of $z^i(y,t)$, obtaining the following equations,

$z^0(y,t)=g(y,t)$, and $z^1(y,t)$ formulate as

$$z^1(y,t)=\lambda\int_a^t\int_a^b G(y,t,v,w)z^0(v,w)dvdw \qquad (4)$$

by integrating we obtain, then $z^2(y,t)$ formulate as

$$z^2(y,t)=\lambda\int_a^t\int_a^b G(y,t,v,w)z^1(v,w)dvdw \qquad (5)$$

The general formula of iterative method is

$$z^{n+1}(y,t)=\lambda\int_a^t\int_a^b K(y,t,v,w)z^n(v,w)dvdw \qquad (6)$$

the numerical solutions which generate by this method $z(y,t)=\sum_{i=0}^{n} z^i(y,t)$ approach to the exact solution of " (1)" .

Theorem (1)

If $g(y,t)$ is bounded function on the connected rejoin $H=\{(y,t);a\le y\le b,a\le t\le k_1\}$ and $G(y,t,v,w)$ is bounded function on the rejoin $F=\{(y,t,v,w);a\le v\le y\le b,a\le w\le t\le k_1\}$, then the sequence $\{z^n(y,t)\}_{n=0}^{\infty}$ generated by the given formula in "(6)"

$$z^{n+1}(y,t)=\lambda\int_a^t\int_a^b G(y,t,v,w)z^n(v,w)dvdw$$

is convergence uniformly to exact solution $z(y,t)$.

Proof: - By suppose $g(y,t)$ and $G(y,t,v,w)$ are bounded then there exist a positive numbers $m\ge 0$ and $d\ge 0$ such that $\|g(y,t)\|\le m$ and $\|G(y,t,v,w)\|\le h$ respectively. Now $z^0(y,t)=g(y,t)$, then $\|z^0(y,t)\|=\|g(y,t)\|\le m$

Since $z^1(y,t)=\lambda\int_a^t\int_a^b G^1(y,t,v,w)g(v,w)dvdw$

$$\|z^1(y,t)\|=\left\|\lambda\int_a^t\int_a^b G^1(y,t,v,w)g(v,w)dvdw\right\|=|\lambda|\int_a^t\int_a^b\|G^1(y,t,v,w)\|\|g(v,w)\|dvdw$$

$$\|z^1(y,t)\|\le|\lambda|\int_a^t\int_a^b(h)(m)dvdw=|\lambda|hm\int_a^t\int_a^b dvdw=|\lambda|hm(b-a)t$$

Since we have

$$z^2(y,t)=\lambda^2\int_a^t\int_\Omega G^2(y,t,v,w)g(v,w)dvdw,$$

then $\|z^2(y,t)\|=\left\|\lambda\int_a^t\int_a^b G^1(y,t,v,w)z^1(v,w)dvdw\right\|$

$$=|\lambda|\int_a^t\int_a^b\|G^1(y,t,v,w)\|(\lambda h(b-a)w)dvdw$$

$$\|z^2(y,t)\|=\lambda\int_a^t\int_a^b h(\lambda mh(b-a)s)dvdw=\lambda^2mh^2(b-a)\int_a^t\int_a^b wdvdw$$

$$=\lambda^2mh^2(b-a)[\frac{(b-a)w^2}{2}]=\frac{\lambda^2mh^2(b-a)^2w^2}{2}$$

$$\|z^3(y,t)\|=\lambda\int_a^t\int_a^b h(\frac{\lambda^2mh^2(b-a)^2w^2}{2})dvdw=\frac{\lambda^3mh^3(b-a)^2}{2}\int_a^t\int_a^b w^2dvdw$$

$$=\lambda^3mh^3(b-a)[\frac{(b-a)w^3}{3!}]=\frac{\lambda^2mh^2(b-a)^2w^3}{3!} \qquad (7)$$

And so on, in general we obtain

$$\|z^n(y,t)\|=\left\|\lambda\int_a^t\int_a^b G(y,t,v,w)z^{n-1}(v,w)dvdw\right\|$$

$$=|\lambda|\int_a^t\int_a^b h(\frac{\lambda^{n-1}mh^{n-1}(b-a)^{n-1}w^{n-1}}{(n-1)!})dvdw$$

$$\|z^n(y,t)\|=\frac{\lambda^nmh^n(b-a)^{n-1}w^{n-1}}{(n-1)!}[(b-a)w]=\frac{\lambda^nmh^n(b-a)^nw^n}{(n)!}$$

By using ratio test [6], we want to show that the series $\|z(y,t)\|=\sum_{i=0}^{n}\|z^i(y,t)\|=\sum_{i=0}^{\infty}\frac{\lambda^imh^i(b-a)^iw^i}{(i)!}$ is convergence. Where the general term is $R_i=\frac{\lambda^imh^i(b-a)^iw^i}{(i)!}$

$$\lim_{i\to\infty}\frac{R_{i+1}}{R_i}=\lim_{i\to\infty}\frac{\frac{\lambda^{i+1}mh^{i+1}(b-a)^{i+1}w^{i+1}}{(i+1)!}}{\frac{\lambda^imh^i(b-a)^iw^i}{(i)!}}=\lim_{i\to\infty}\frac{\lambda h(b-a)w}{(i+1)}=0 \qquad (8)$$

Therefore, it is convergent for all values of $\lambda,h,(b-a)$ and k_1 the series is absolutely and uniformly convergent $\forall,(y,t)\in H.$

Theorem (2) [6]
Let $z(y,t)$ be a smooth function and $z^n(y,t)$ be the nth approximate solution of $z(y,t)$ then $\|z(y,t)-z^n(y,t)\| \le \frac{\lambda E}{(2\pi)^n}$, where E is a positive constant independent of n and is a bound for the partial derivative of $z(y,t)$, such that $\pi = 3.1428$.

Theorem (3)
Supposes that $z(y,t)$ be the exact solution of TDIMVFIE, $g(y,t)$ is bounded function on $H=\{(y,t); a \le y \le b, a \le t \le k_1\}$ and $G(y,t,v,w)$ is bounded function on the rejoin $F=\{(y,t,v,w); a \le v \le y \le b, a \le w \le t \le k_1\}$, then a mapping $L: M \to N$ is contraction mapping.

Proof: - The nth approximate solution has the form which is given in "(6)"

$$z^n(y,t) = g(y,t) + \lambda \int_a^t \int_a^b G(y,t,v,w) z^{n-1}(v,w)\,dvdw$$

Then we want to prove that L is contraction mapping for sufficient large n,

$$\|L(z(y,t)) - L(z^n(y,t))\| = \Big\| g(y,t) + \lambda \int_a^t \int_a^b G(y,t,v,w) z(v,w)\,dvdw$$

$$-[g(y,t) + \lambda \int_a^t \int_a^b G(y,t,v,w) z^{n-1}(v,w)\,dvdw \Big\|$$

$$= \Big\| \lambda \int_a^t \int_a^b G(y,t,v,w)(z(v,w) - z^{n-1}(v,w))\,dvdw \Big\|$$

$$\le \lambda \int_a^t \int_a^b \|G(y,t,v,w)\| \|z(v,w) - z^{n-1}(v,w)\|\,dvdw$$

Since $G(y,t,v,w)$ is bounded function on d then $\|G(y,t,v,w)\| \le d$

$$= \Big\| \lambda \int_a^t \int_a^b d[z(v,w) - z^{n-1}(v,w)]\,dvdw \Big\| \le \lambda d \int_a^t \int_a^b \|z(v,w) - z^{n-1}(v,w)\|\,dvdw$$

By using theorem (2), we have

$$\|z(y,t) - z^{n-1}(y,t)\| \le \frac{\lambda E}{(2\pi)^{n-1}}$$

$$\le \lambda d \int_a^t \int_\Omega \frac{\lambda E}{(2\pi)^{n-1}}\,dvdw \le \lambda d \frac{\lambda E}{(2\pi)^{n-1}} \int_a^t \int_a^b dvdw$$

$$\|L(z(y,t)) - L(z^n(y,t))\| \le \frac{\lambda^2 EM}{(2\pi)^{n-1}} \int_a^t \int_a^b dvdw \quad (9)$$

Then for $n \to \infty$, we get $\frac{\lambda^2 Ed}{(2\pi)^{n-1}} \int_a^t \int_a^b dvdw \to 0$, Therefor

$$\|L(z(y,t)) - L(z^n(y,t))\| \to 0 \quad (10)$$

Hence, L is contraction mapping.

Algorithm of ADM

Input: a, b, n, w, k_1, Er

Step 1: Suppose that the numerical solution of TDIMVFIE be

$$z(y,t) = \sum_{i=0}^{n} z^i(y,t)$$

Step 2: Put $z^0(y,t) = g(y,t)$

For $i = 0$ to n

Step 3: Finding the component $z^i(y,t)$, by using " (6)" .

Step 4: Computing the absolute error by using this formula

$$r^n = \Big\| \sum_{i=0}^{n} z^i(y,t) - z(y,t) \Big\|.$$

Step 5: If $r^n < Er$ then go to output.

End if

End for

Step 6: Continuous in this process to get the approximate solution of the problem.

Output: the results of approximate solution and r^n .

III. NUMERICAL EXAMPLES

In this section, displaying some numerical examples and discussing the results of solving example which obtained by new cooperating technique.

Example 1: - Find approximate solution of TDIMVFIE

$$z(y,t) = 4y + 2t - y\frac{(y-t)}{6}(-8y^2 + 3y + 4) + \int_0^t \int_0^1 (y-t)(v-w) z(v,w)\,dvdw$$

where the exact solutions $z(y,t) = 4y + 2t$.

Solution. Applying the ADM on this problem, obtaining the following results.

TABLE I. COMPARISON BETWEEN THE RESULTS.

The point (y,t)	Number of iterations n	Exact solution of $z(y,t)$	Approximate solutions by ADM.	Absolute error
(0.1,0.2)	1 3 5 7 9	0.800000	0.79938888889 0.79999366567 0.79999993886 0.79999999940 0.79999999999	6.1111×10^{-4} 6.3343×10^{-6} 6.1143×10^{-8} 5.9040×10^{-10} 5.7012×10^{-12}

TABLE II: SHOWS LSE AND ACCORDING TO THE NUMBER OF ITERATIONS

Number of iterations	LSE	RT

1	4.4673×10^{-6}	0:0:1.2632
3	3.6754×10^{-9}	0:0:2.3491
5	6.2318×10^{-12}	0:0:4.6793
7	2.5769×10^{-15}	0:0:6.9841

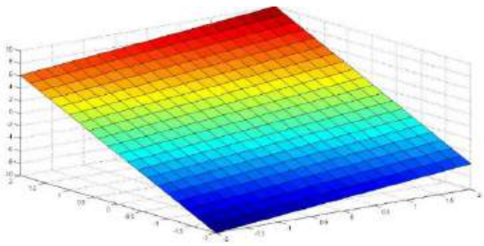

Fig. 1: Plot of $z(y,t)=\sum_{i=0}^{2} z^i(y,t)$approximation by ADM.

Example 2:- Find approximate solution of TDIMVFIE-2 [12]

$$z(y,t)=y^2+3t-yt^2\frac{(3y^3+2y^2+9t+12)}{12}+\int_0^t\int_0^1 yt(v-w)z(v,w)dvdw$$

where the exact solutions $z(y,t)=y^2+3t$.

Solution. Applying the ADM on this problem, obtaining the following results.

TABLE III : COMPARISON BETWEEN THE RESULTS

The point (y,t)	Number of iterations n	Exact solution of $z(y,t)$	Approximate solutions by ADM.	Absolute error
(0.1,0.2)	1	0.610000	0.609782825	2.1717×10^{-4}
	3		0.609993774	6.2251×10^{-6}
	5		0.609999823	1.7636×10^{-7}
	7		0.609999995	4.9964×10^{-9}
	9		0.609999999	1.4155×10^{-11}

TABLE IV: SHOWS LSE and RT ACCORDING TO THE NUMBER OF ITERATIONS

Number of iterations	LSE	RT
1	2.5532×10^{-5}	0:0:1.9043
3	4.7614×10^{-8}	0:0:2.8654
5	3.6802×10^{-12}	0:0:4.7951
7	6.3267×10^{-14}	0:0:6.9553

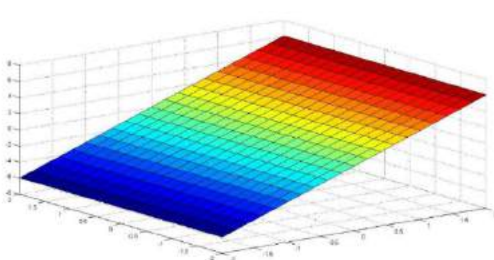

Fig. 2: Plot of $z(y,t)=\sum_{i=0}^{2} z^i(y,t)$ approximation by ADM.

IV. CONCLUSIONS

In this work, reformulating ADM for finding the numerical solutions TDIMVFIE. Proving the convergence of the technique. Applying ADM achieved numerical solution and gives very good results. Tables I and III showed the comparison results of different iterations. For the criterion of discussion calculating the LSE and RT which revealed in tables II and IV . Finally, form the results demonstrated results we conclude that, the ADM is extremely effective based on the comparisons results and shows the high accuracy of ADM and its appropriateness.

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