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Research Article

Modification of Adomian Decomposition Method to Solve Two Dimensions Volterra Integral Equation Second Kind

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ABSTRACT

In this work, the adomian decomposition method is proposed for the first time to solve problems in two-dimensional Volterra Integral equations of the second kind. In addition, modification of this method is introduced to address this problem. Several new theorems are introduced and proven regarding contractive mapping and uniform convergence to support the method. Numerical problems are solved using the software Matlab to obtain results and demonstrate the efficiency and simplicity of the technique. The modified Adomian decomposition method performs faster than the adomian decomposition method for obtaining results.

Keywords: Adomian decomposition method (ADM), two-dimensional Volterra Integral equations numerical solution, least square error, uniformly convergence

INTRODUCTION

Integral equations are a branch of mathematics that plays a significant role in various fields. Over recent decades, many problems involving integral equations have been formulated from different scenarios in applied sciences such as mathematical physics, engineering, and electromagnetic waves [1], [3], [7]. Varies problems have evolved in two – dimensional (2D) integral equations, with one type being the 2D Volterra equation of the second kind (TDVIE-2^{nd}).

The TDVIE-2^{nd} arises in various phenomena, physics, and engineering areas as the Electric field integral equation, and partial differentiation equations defined in closed regions can be transformed into this type [5], [6], [9]. Usually, finding an analytic solution for TDVIE-2 is very difficult. Therefore, to address these problems, we must find the numerical solution for it**.** In scientific research of applied sciences, the numerical method is an important tool for treating integral equations [10], [12], [13], 14].

Another example is the use of numerical techniques for analyzing the radiation of dipole antenna. Evaluating radiation patterns for TD segments in three-dimensional (3D) patterns [2], [13].

The numerical method was used to estimate electromagnetically features for element-loaded media such as electromagnetic waves and sea ice. An example in physics is the solid angle beam of an antenna, which is described in equation (1) showing the integral for normalizing power patterns on a sphere (4π, s, r). The mathematical representation of it is given by equations [8], [16]:

$$\Omega A = \int_0^{2\pi}\int_0^{\pi} P(\emptyset,\Theta)\sin(\emptyset)d\emptyset d\Theta) \quad \text{and}$$

$$\Omega A = \int\int_{4\pi}^{s\,r} P(\emptyset,\Theta)d\Omega \tag{1}$$

Polar coordinates increments are displayed. Solid angle dA = r^2dΩ over the surface of a sphere of radius r, where dΩ was the solid angle that was subtended by area.

DEFINITIONS AND THEOREMS

Introducing some definitions and theorems.

Definition 1. The following equation

$$u(e,h) = w(e,h) + \tau\int_a^h\int_c^e G(e,h,y,z)u(y,z))dydz. \tag{2}$$

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Is named TDVIE-2nd, such that w(e,h) and G(e,h,y,z) ≠ 0 were given continuous functions on region $\mathfrak{S}=\{(e,h):0\le h\le c_1,\ a\le e\le b\}$, and $\mathfrak{P}=\{(e,h,y,z):a\le y\le e\le b, 0\le z\le h\le c_1\}$ respectively, τ is a scalar while u(x,t) is the unknown functions.[4,14]

SOLVING TDVIE-2ND BY ADM.

Here, reformulating and applying ADM to solve TDVIE-2nd, let

$$u(e,h)=\sum_{i=0}^{n}u^{i}(e,h) \tag{3}$$

be the numerical treatment for equation (2), then substituting it in equation (1) we get

$$\sum_{i=0}^{n}u^{i}(e,h)=w(e,h)+\tau\int_a^t\int_c^x G(e,h,y,z)\sum_{i=0}^{n}u^{i}(y,z)\,dydz. \tag{4}$$

After those powers for u^i (e,h) equaling, we get

u^0 (e,h) = w(e,h) is initial solution. Now u^1 (x,t) formulate as

$$u^{1}(e,h)=w(e,h)+\tau\int_a^h\int_c^e G(e,h,y,z)u^{0}(y,z))dydz.$$

Integrating it for variables. Then u^2 (e,h) formulate as

$$u^{2}(e,h)=w(e,h)+\tau\int_a^h\int_c^e G(e,h,y,z)u^{1}(y,z))dydz.$$

Then the second formula of iterative. We find u^3 (e,h) by using the form, as

$$u^{3}(e,h)=w(e,h)+\tau\int_a^h\int_c^e G(e,h,y,z)u^{2}(y,z))dydz.$$

And the general formula of the iterative method is

$$u^{n}(e,h)=w(e,h)+\tau\int_a^h\int_c^e G(e,h,y,z)u^{n-1}(y,z))dydz. \tag{5}$$

After computing these components, we get the numerical solution for TDVIE-2nd as

$$u(e,h)=\sum_{i=0}^{n}u^{i}(e,h)$$

Note that, we chose Y^0 = u^0 (e,h) = w(e,h), and the iteration

$$Y^{n}=T(Y^{n-1}) \tag{6}$$

Where $Y^{n}=u^{n}(e,h)$ and

$$T(Y^{n})=\tau\int_a^h\int_c^e G(e,h,y,z)u^{n-1}(y,z))dydz.$$

Theorem 1: Let the smooth function u (e,h) and u^n(e,h) is nth numerical solution for z (x,t) then $u(e,h)-u^{n}(e,h)\le\frac{\lambda E}{(2\pi)^{n}}$, where E > 0 independence on n and bounded by partial derivatives of u(e,h) where π = 3.14287 [14].

Theorem 2: Assume u (e,h) is the actual solution for TDVIE-2nd, such that w(e,h) defined on regions $\mathfrak{S}=\{(e,h):a\le h\le b,\ c\le e\le c_1\}$ and G (e,h,y,z) on $\mathfrak{P}=\{(e,h,y,z):a\le h\le e\le b, c\le z\le h\le c_1\}$ are bounded functions, then $L:M\to N$ is contractive mapping.

Proof: The nth numerical solution defined in equation (5) as,

$$u^{n}(e,h)=\tau\int_a^h\int_c^e G(e,h,y,z)u^{n-1}(y,z))dydz.$$

Now

$$L(u(e,h))-L(u^{n}(e,h))=\tau\int_a^h\int_c^e G(e,h,y,z)u(y,z))dydz$$

$$-\tau\int_a^h\int_c^e G(e,h,y,z)u^{n-1}(y,z))dydz$$

$$=\tau\int_a^h\int_c^e G(e,h,y,z)(u(y,z)-u^{n-1}(y,z))dydz$$

$$\le\tau\int_a^h\int_c^e G(e,h,y,z)u(y,z)-u^{n-1}(y,z))dydz$$

Since G(e, h, y, z) bounded function then G(e, h, y, z ≤ d)

$$\le\tau\int_a^h\int_c^e d(u(y,z)-u^{n-1}(y,z))dydz\le\tau d\int_a^h\int_c^e(u(y,z)-u^{n-1}(y,z))dydz$$

By using theorem (1), getting

$$u(e,h)-u^{n-1}(e,h)\le\frac{\tau E}{(2\pi)^{n-1}}$$

$$\le\lambda d\int_a^h\int_c^e\left(\frac{\tau E}{(2\pi)^{n-1}}\right)dydz=\lambda d\frac{\tau E}{(2\pi)^{n-1}}\int_a^h\int_c^e dydz$$

$$L(u(e,h))-L(u^{n}(e,h))\le\tau d\frac{\tau E}{(2\pi)^{n-1}}\int_a^h\int_c^e dydz$$

Then for $n\to\infty$, we get $\tau d\frac{\lambda E}{(2\pi)^{n-1}}\int_a^h\int_c^e dydz\to 0$,

Therefore L(u(e,h)) − $L(u(e,h))-L(u^{n}(e,h))\to 0$.

Note: E_r is the error and a,t,n,e,k_1 are constants.

Algorithm steps of the technique.

Input: a,t,n,e,Er,k_1

1. Let $u(e,h)=\sum_{i=0}^{n}u^{i}(e,h)$ be the approximate solution for TDVIE-2nd.

2. Put $u^0(\mathrm{e,h}) = \mathrm{w(e,h)}$

for $i = 1$ to n

3. In equation (5) calculate u^i (e,h).

4. Absolute error calculating by $r^n = \left|\sum_{i=0}^{n} u^i(e,h) - u(e,h)\right|$.

If $r^n \le E_r$ Go to output

End if

End for

5. Continues in this way, till obtaining the numerical solution for TDVIE-2nd.

Output: Numerical results and r^n.

Theorem 3: The TDVIE-2nd satisfies these conditions [13].

1. G(e, h, y, z) is real and continuous function, and |G(e, h,y,z)| ≤ m in

$\mathfrak{P} = \{(e,h,y,z) : a \le y \le e \le b, c \le z \le h \le c_1\}$ where G(e, h, y,z) ≠ 0

2. w(e,h) is real and continuous in the region $\mathfrak{S} = \{(e,h) : a \le h \le b,\ c \le e \le c_1\}$, |w(e,h)| ≤ m in $\mathfrak{S}$ and w (e,h) ≠ 0. Then it has one and only one continuous solution u (e,h) in $\mathfrak{S}$.

Theorem 4: Let u (x,t) be a function defined over $\mathfrak{S} = \{(e,h) : a \le h \le b,\ a \le e \le t\}$, G (e, h, y, z) be a continuous function over the region $\mathfrak{P} = \{(e,h,y,z) : a \le y \le e \le t, c \le z \le h \le c_1\}$ where |G(e, h, y, z) ≤ M|, M is positive constant, w(e,h) is a continuous function on $\mathfrak{S}$ where|w(e,h) ≤ m|, then $\{u_n$ (e,h)} given as

$$u_n(e,h) = \tau \int_a^h \int_c^e G(e,h,y,z) u_{n-1}(y,z) dydz,$$

with u_0 (e,h) = w(e,h) is uniformly convergent to u(x,t)

Proof: Since u_0 (e,h) = w (e,h), then

$|u_0$ (e,h)| = |w(e,h)| ≤ m, and

$$|u_1(e,h)| = \left|\tau \int_a^h \int_c^e G(e,h,y,z) u_0(y,z) dydz\right|$$

$$\le |\lambda| mM(h-a)(e-c)$$

Also,

$$|u_2(e,h)| = \left|\tau \int_a^h \int_c^e G(e,h,y,z) u_1(y,z) dydz\right|$$

$$\le |\tau| \int_a^h \int_c^e |G(e,h,y,z)| mM(y-a)(z-c) dydz$$

$$\le |\lambda|^2 mM^2 \left(\frac{1}{2*2}(h-a)^2(e-c)^2\right)$$

and

$$|u_3(e,h)| = \left|\tau \int_a^h \int_c^e G(e,h,y,z) u_2(y,z) dydz\right|$$

$$\le |\tau| \int_a^h \int_c^e |G(e,h,y,z)| \left(\frac{1}{2*2}(y-a)^2(z-c)^2\right) dydz$$

$$\le |\tau|^3 mM^3 \left(\frac{1}{2^2 3^2}(h-a)^3(e-a)^3\right)$$

Carrying in a similar manner, we get $|u_n(e,h)| \le \frac{m}{2^2 3^2 4^2 \ldots n^2} |\tau|^n M^n (h-a)^n (e-c)^n]$

As $n \to \infty$ we have

$$|u_\infty(x,t)| \le \lim_{k\to\infty} \frac{m}{2^2 3^2 4^2 \ldots k^2} |\tau|^k M^k (c_1 - a)^k (c_2 - c)^k$$

By using the ratio test.

$$\lim_{k\to\infty} \frac{R_{n+1}}{R_n}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{2^2 3^2 4^2 \ldots k^2 (k+1)^2} |\tau|^{k+1} M^{k+1} (c_1 - a)^{k+1} (c_2 - c)^{k+1}.}{\frac{1}{2^2 3^2 4^2 \ldots k^2} |\tau|^k M^k (c_1 - a)^k (c_2 - c)^k.}$$

$$= \lim_{k\to\infty} \frac{|\lambda| M((c_1 - a)(c_2 - c))}{(k+1)^2} = 0.$$

Then, it is convergence ∀ of τ, M, m, $(c_1 - a)$ and $(c_2 - c)$

Hence, it is absolutely and uniformly convergent $\forall (e,h) \in \mathfrak{S}$.

SOME EXAMPLES

In integral equation problems, we often make approximations of the unknown function to facilitate numerical computations and obtain solutions. This approach also helps in writing programs for implementing these techniques on a computer. Discussing the use of ADM for solving TDVIE-2nd through numerical examples.

Example 1. Find the approximate solution of TDVIE-2nd.

$$u(e,h) = eh - \frac{h^5 e^4}{9} + \int_0^h \int_0^e (eh^2 yz) u(y,z) dydz.$$

where the exact solutions u(e,h) = eh.

Solution: Using ADM, put

$$u^0(e,h) = w(e,h) = eh - \frac{h^5 e^4}{9}$$

Hence, the first and second approximations can be obtained

$$u^1(e,h) = -\frac{h^5 e^4 (h^3 e^4 - 42)}{378}$$

$$u^2(e,h) = -\frac{h^8e^8(h^4e^3-100)}{3762}$$

Note that after two iterations the absolute error is zero.

$(e,h)=(0.1,0.2)$

Example 2. Find the approximate solution of TDVIE-2nd.

$$u(e,h) = e^2+h^2-\frac{he(h+e)^2(3h^2-he-3e^2)}{12}$$
$$+\int_0^h\int_0^e (e+h)(y+z)u(y,z)\,dydz.$$

With the exact solution $u(e,h) = e^2 + h^2$.

Solution: Using ADM, put

$$u^0(e,h) = w(e,h) = e^2+h^2-\frac{he(h+e)^2(3h^2-he-3e^2)}{12}$$

Hence, the first and second approximations obtained,

$$u^1(e,h) = -\frac{he(t+e)^2(270h^5e+290h^4e^2+277h^3e^3)}{15120}$$
$$\frac{+290h^2e^4-3780h^2+270he^5+1260he-3780e^2)}{15120}$$

$$u^2(e,h) = -\frac{h^2e^2(h+e)(3240h^7e+7760h^6e^2+9853h^5e^3-9527h^4e^4)}{5443200}$$
$$+\frac{(97200h^4+9853h^3e^5-104400h^3e+7760h^2e^6-99720t^3x^2)}{5443200}$$

MODIFICATION FOR ADM TO SOLVE TDVIE-2ND

In the modification, ADM assumes that w(e,h) written as

$$w(e,h) = w_1(e,h)+w_2(e,h) \quad (7)$$

for minimizing the steps for computations, therefore fasting the convergence process.

Put u^0 (e,h) = w_1(e,h) and

$$u^1(e,h) = w_2(e,h) + \tau\int_0^h\int_\Omega G(e,h,y,z)u^0(y,z)\,dydz,$$

$$u^{k+1}(e,h) = w_2(e,h) + \tau\int_0^h\int_\Omega G(e,h,y,z)u^k(y,z)\,dydz, \quad (8)$$

If the opposite terms accurse between the functions $u^0(e,h)$ and $u^1(e,h)$, deleting these terms between these two functions, then remain terms in $u_0(e,h)$ probably given the actual solution for the problem.

Example 3: Solve example 1 by modifying ADM.

Solution: Using modified ADM.

since $w(e,h) = eh-\frac{h^5e^4}{9}$, then

Table 1: Comparison of numerical and exact solutions for the ADM in the point (e, h) = (0.1, 0.2) with the exact solution 0.02000000.

Iterations	Approximate solution by ADM	Absolute error for ADM
0	0.0199999644	3.5664×10^{-8}
1	0.0199999983	1.6732×10^{-9}
2	0.0200000000	1.4000×10^{-11}
3	0.0200000000	0.0000000000

Table 2: A comparison of numerical and exact solutions for the ADM in the point (e, h) = (0.1, 0.2) with the exact solution 0.05000000

Iteration	Approximate solution by ADM	Absolute error for ADM
0	0.0798933333	2.9893×10^{-2}
1	0.0499999999	2.0471×10^{-9}
2	0.0499999999	3.6320×10^{-12}
3	0.0500000000	0.0000000000

Table 3: Results of example (1) for different iterations

	First iteration	Third iteration	Fifth iteration	Seventh iteration
ADM				
LSE	4.77×10^{-5}	3.43×10^{-8}	2.65×10^{-12}	6.16×10^{-15}
RT	0:12675	0:17832	0:2.4473	0:3.0542

Table 4: Results of example (2) for different iterations

	First iteration	Third iteration	Fifth iteration	Seventh iteration
ADM				
LSE	2.56×10^{-7}	3.43×10^{-10}	6.67×10^{-11}	3.68×10^{-14}
RT	0:1.6532	0:19564	0:27754	0:3.2243

$w(e,h) = w_1(e,h)+w_2(e,h)$ Put $w_1(e,h)=eh$ and $w_2(e,h) = -\frac{h^5e^4}{9}$, then $u^0(e,h)=w_1(e,h)=eh$,

then first approximation getting by

$$u_1(e,h) = w_2(e,h)+\int_0^h\int_0^e G(e,h,y,z)u_0(y,z)\,dydz$$
$$= -\frac{he^4}{9}+\lambda\int_0^h\int_0^e (eh^2yz)u_0(y,z)\,dydz = -\frac{h^5e^4}{9}$$
$$+\int_0^h\int_0^e (eh^2yz)(yz)\,dydz = 0$$

Then, by computing these two components, we get the exact solution for the problem.

Example 4: Solve example 2 by modifying ADM.

Solution: Using modified ADM.

since $w(e,h) = e^2 + h^2 - \frac{he(h+e)^2(3h^2 - he - 3e^2)}{12}$ then

$w(e,h) = w_1(e,h) + w_2(e,h)$ put $w_1(e,h) = e^2 + h^2$ and $w_2(e,h) = -\frac{he(h+e)^2(3h^2 - he - 3e^2)}{12}$

$u^0(e,h) = w_1(e,h) = e^2 + h^2$, then first approximation getting by

$$u_1(e,h) = w_2(e,h) + \int_0^h \int_0^e G(e,h,y,z)\, u_0(y,z)\, dydz$$

$$= -\frac{he(h+e)^2(3h^2 - he - 3e^2)}{12} + \int_0^h \int_0^e (e+h)(y+z)\left(y^2 + z^2\right) dydz = 0$$

Then by computing these two components, we get the exact solution for the problem.

CONCLUSIONS

This study proposes numerical solutions for TDVIE-2^{nd} utilizing the ADM method, suggesting suitable algorithms, and presenting results through a Matlab software program. The numerical results were compared based on LSE and RT to facilitate a discussion. Finally, we may infer that:

1. The ADM demonstrated their effectiveness for numerically solving TDVIE-2^{nd} and producing precise answers.
2. From tables (3) and (4) the LSE decreases when the number of iterations increases.
3. Modified ADM needs only one iteration for getting the exact solution for the problem.

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